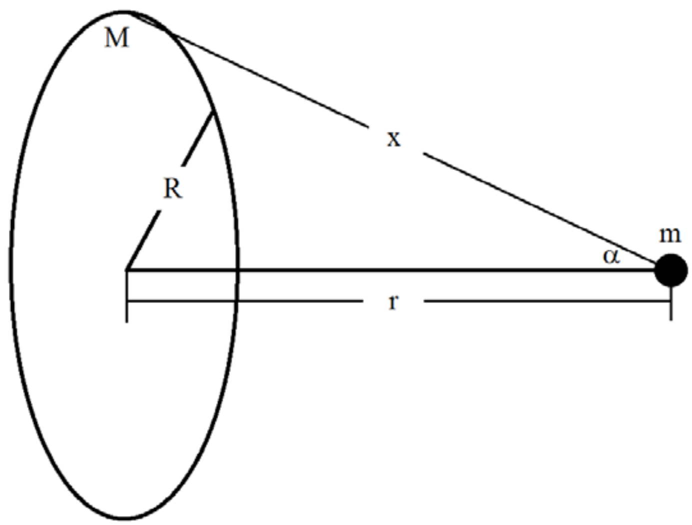
Takashi's Institute of Technology

Special Lecture Notes

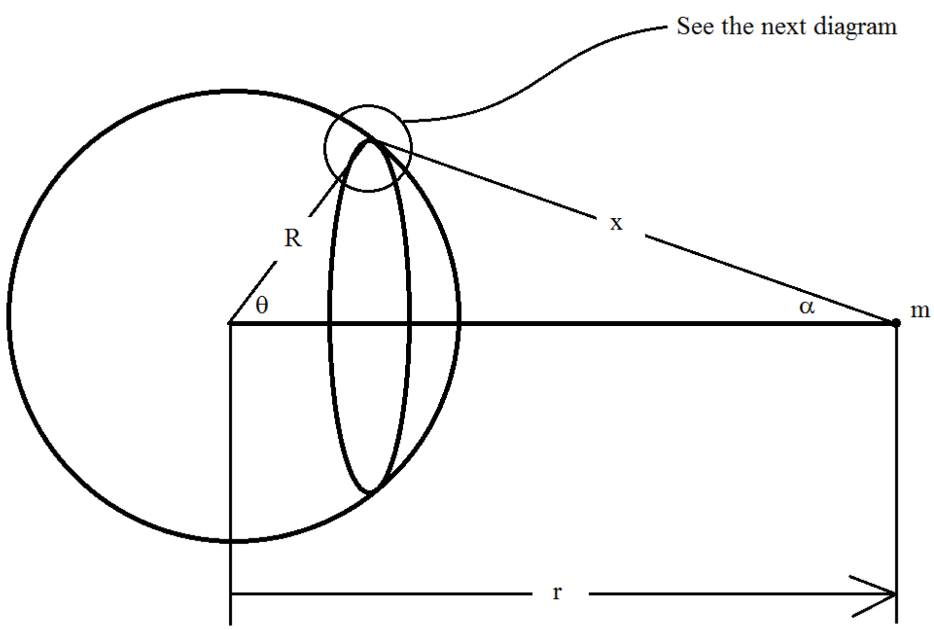
Derivation of Gravitational force of a hollow sphere

The setup: There are two objects, a hollow sphere (of mass "M", radius "R", and thickness "t",) and a point mass of "m".

Just like others, we are going to take a small chunk (a ring in this case), and calculate the net gravitational force, dF, acting on m by the ring. A reader should be able to prove that the gravitational force created by a ring onto m is $F_{grav} = G\frac{Mm}{x^2}\cos\alpha$.

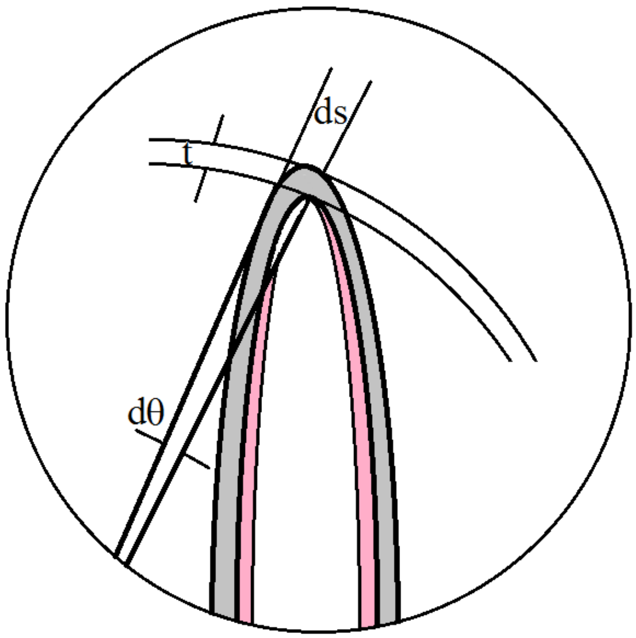


After proving this, the gravitational force acting on "m" by a ringlet is calculated as $dF_{grav} = G\frac{dm \cdot m}{x^2}\cos\alpha$, where "dm" is the mass of the ringlet and the direction of dF is from "m" to the center of the sphere.



To calculate "dm", we need to know "ρ" and "$d(vol)_{ringlet}$".

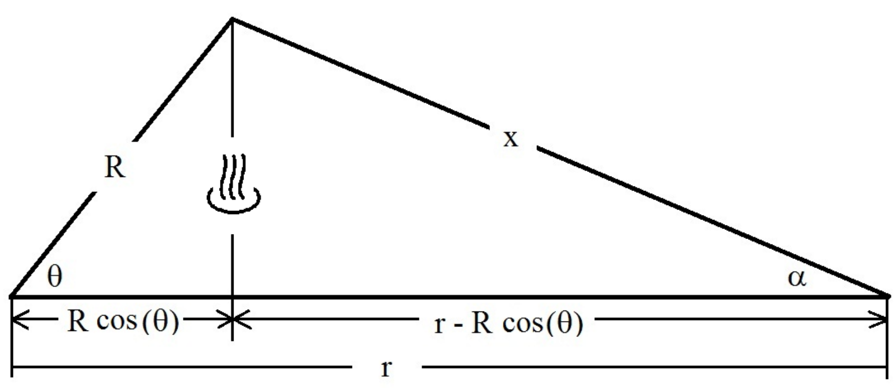
$\rho = \frac{\text{Mass}}{\text{Voume}} = \frac{M}{4\pi R^2 t}$, and $d(vol)_{ringlet} = 2\pi 𝕣^2 \cdot ds \cdot t$, where "𝕣" is the radius of the ringlet (this is the third r and this is the reason why we need a "bunny", but I can't find it on a computer), "ds" is the width of the ringlet, and "t" is the thickness of the ringlet (as well as the thickness of the sphere). Notice that we can't use "r" for the radius of the ringlet since "r" has been used as the distance between the center of the sphere to the point mass, "m".



$$\therefore dF_{grav} = G\frac{dm \cdot m}{x^2}\cos\alpha = G\frac{\rho \cdot d(vol) \cdot m}{x^2}\cos\alpha = G\frac{\rho \cdot (2\pi ♨) \cdot ds \cdot t \cdot m}{x^2}\cos\alpha$$

At this point, there are four variables, x, ♨, ds, and α. We will try to express dF as a function of one variable "x".

First, let's focus on the triangle



From the diagram, you can tell that:

$\cos\alpha = \frac{r - R\cos(\theta)}{x}$

$♨ = R\sin(\theta)$

And $ds = R \cdot d\theta$

Hence,

$$dF_{grav} = G\frac{\rho\cdot(2\pi)\cdot(R\sin(\theta))\cdot(R\cdot d\theta)\cdot t\cdot m}{x^2}\cdot\frac{(r - R\cos(\theta))}{x}$$

$$= G\rho 2\pi R^2 tm\frac{(\sin(\theta))\cdot(r - R\cos(\theta))}{x^3}\cdot d\theta$$

In handwriting, $G\rho 2\pi R^2 tm$ is a “jellyfish”. In here, we will use $G\rho 2\pi R^2 tm \equiv$ ♘.

We still have two variables, “x” and “θ”. As described before, dF_{grav} is going to be a function of “x” only. To change from “θ” to “x”, we use “Law of cosine” (this is the reason that you were asked to be able to derive at the beginning of the semester.) If you did not learn then, here is the time to learn:

In the previous diagram, notice that "♨" is shared by the left and right triangle.

<u>Left Hand Side</u> <u>Right Hand Side</u>

$$R^2 - (R\cos(\theta))^2 = \text{♨}^2 = x^2 - (r - (R\cos(\theta))^2$$

Equating the both sides, you should be able to end up with

$$R^2 = x^2 - r^2 + 2rR\cos(\theta)$$

$$\therefore\ r\cos(\theta) = \frac{R^2 - x^2 + r^2}{2R}$$

Also, sin(θ) d(θ) can be derived by differentiating the equation above (after dividing both sides by “r”.)

$$\cos(\theta) = \frac{R^2 - x^2 + r^2}{2Rr}\ \text{("x" is the only variable on this side)}$$

$$-\sin(\theta)\cdot d\theta = \frac{-2x}{2Rr}\cdot dx$$

$$\sin(\theta)\cdot d\theta = \frac{x}{Rr}\cdot dx$$

$$dF_{grav} = \text{♘}\,\frac{(r - R\,\frac{R^2 - x^2 + r^2}{2Rr})}{x^3}\cdot(\frac{x}{Rr}\cdot dx)$$

$$= \text{♘}\,\frac{(r - \frac{R^2 - x^2 + r^2}{2r})}{x^2}\cdot\frac{1}{Rr}\cdot dx$$

$$= \text{♘} \frac{\left(\frac{2r^2 - (R^2 - x^2 + r^2)}{2r}\right)}{x^2 \cdot Rr} \cdot dx$$

$$= \text{♘} \frac{(2r^2 - R^2 + x^2 - r^2)}{2r^2Rx^2} \cdot dx$$

$$= \text{♘} \frac{(r^2 - R^2 + x^2)}{2r^2Rx^2} \cdot dx$$

$$= \frac{\text{♘}}{2r^2R} \left(\frac{(r^2 - R^2)}{x^2} + 1\right) \cdot dx$$

Finally, dF_{grav} is a function of "x" alone. Now, it is ready to be integrated, but what are the limits? The shape of dF is a vertical ring and "x" is measured from the ring to "m". Therefore the lower limit is at the closest point, the right side of the sphere. Then, x = r – R and the upper limit is the furthest point from "m" to the sphere, x = r + R.

$$F_{grav} = \int dF_{grav} = \int_{r\,-\,R}^{r\,+\,R} \frac{\text{♘}}{2r^2R} \left(\frac{(r^2 - R^2)}{x^2} + 1\right) \cdot dx$$

$= \frac{\text{♘}}{2r^2R} \left(- \frac{r^2 - R^2}{x} + x\right) I_{r\,-\,R}^{r\,+\,R}$ (No, this is not an "I". I could not find a vertical bar to indicate limits!)

$$= \frac{\text{♘}}{2r^2R} \left[\left(- \frac{r^2 - R^2}{(r + R)} + (r + R)\right) - \left(- \frac{r^2 - R^2}{(r - R)} + (r - R)\right)\right]$$

$$= \frac{\text{♘}}{2r^2R} [(- (r - R) + (r + R)) - (- (r + R) + (r - R)]$$

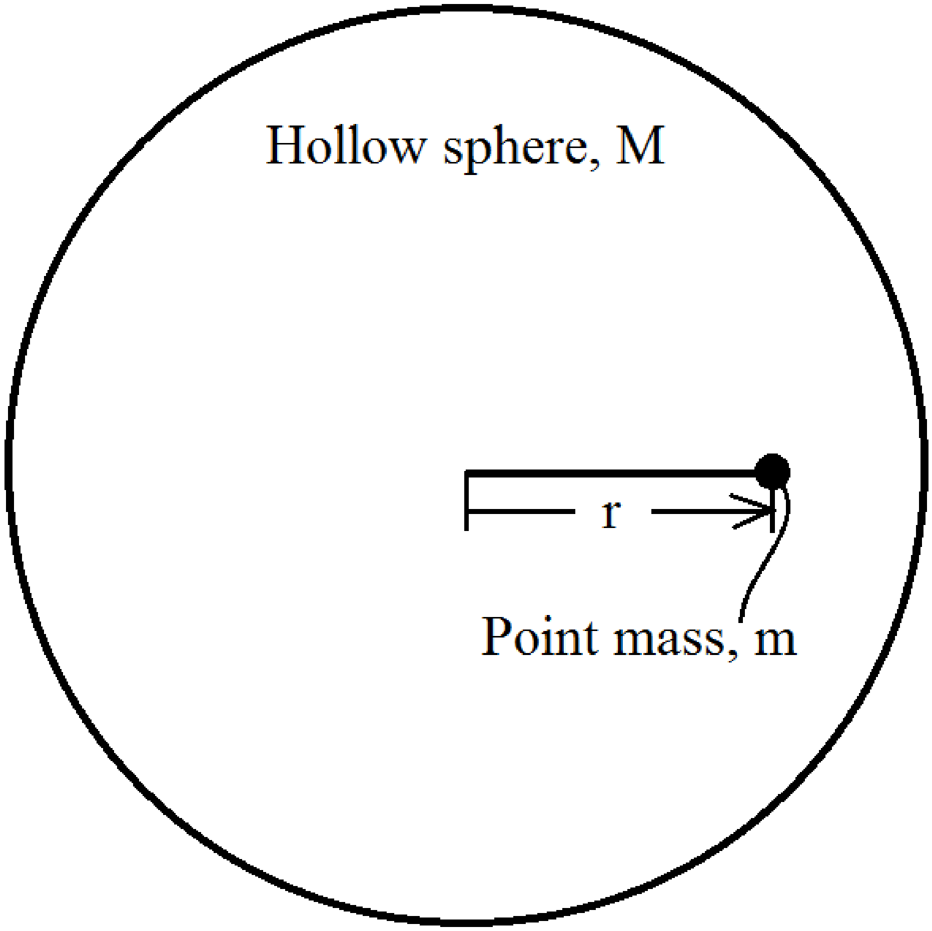
$$= \frac{\text{♘}}{2r^2R} [4R]$$

$$= \frac{2\text{♘}}{r^2} = \frac{2}{r^2} G\rho 2\pi R^2 tm = \frac{2}{r^2} G \left(\frac{M}{4\pi R^2 t}\right) 2\pi R^2 tm$$

$= G\frac{Mm}{r^2}$, which, of course, is the expected answer.

If we can use the distance from the center of mass of a hollow sphere to "m", you should be able to show that the gravitational force acting on "m" by a solid sphere as well.

You should also try to find the gravitational force acting on "m" when it is inside of a hollow sphere. I hope you will notice that the setup for this is exactly the same way as the mass, "m" when it is outside. The only difference between the two cases is the "limits". The result in this case is amazing.


Hollow sphere, M
r
Point mass, m